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CLASSIFICATION OF INTEGRALS OF THE DYNAMIC EQUATIONS OF THE LINEAR TWO-DIMENSIONAL THEORY OF SHELLS

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A known classification is established for the static equations of linear shell theory. Integrals therein corresponding to the membrane, pure bending state of stress, edge effects, state of stress with high variability, etc., are distinguished. These concepts can underlie all known approximate methods of shell statics [1].

The same classification is established herein for the integrals of the two-dimensional dynamic equations of linear shell theory. Appropriate modifications of the approximate equations are deduced. It is shown that a more subdivided classification is expedient in dynamics, in which the variability of the desired state of stress must be taken into account not only in the geometric variables, but also in time.

1. Let us refer the shell middle surface to an arbitrary orthogonal coordinate system with parameters α_1, α_2. Let A_1, A_2 denote the coefficients of the first quadratic form, R_{11}, R_{12}, R_{22}, u_1, u_2 the radii of surface curvature and the displacements in the coordinate directions, and m the mass per unit area of the middle surface. We borrow the remaining notation from the monograph [1]. Also taken therefrom are the general equations of shell theory (with the addition of internal terms and the above-mentioned partial change in the notation).

Let us introduce the small parameter η defined by the equality

$$\eta = h / r \tag{1.1}$$

in which h is half the shell thickness, and r is the characteristic radius of curvature of the middle surface, and let us replace the independent variables and the desired quantities by means of the formulas

$$\begin{array}{lll}
\alpha_i = \eta^a \xi_i, & t = \sqrt{\dfrac{m(1-\sigma^2)}{2Eh}}\,\eta^b \tau & \\
u_i^* = \eta^{-c} u_i, & w^* = w & \\
\varepsilon_i^* = \eta^{a-c}\, \varepsilon_i, & \omega^* = \eta^{a-c}\, \omega, & \delta^* = \eta^{a-c}\, \delta \\
\omega_i^* = \eta^{a-c}\, \omega_i, & \gamma_i^* = \eta^a \gamma_i & \\
T_i^* = \eta^{a-c}\, T_i, & S^* = \eta^{a-c}\, S & \\
\varkappa_i^* = \eta^{2a} \varkappa_i, & \tau^* = \eta^{2a} \tau & \\
G_i^* = \eta^{2a-2} G_i, & H^* = \eta^{2a-2} H, & N_i^* = \eta^{3a-2} N_i
\end{array} \tag{1.2}$$

The numbers a, b, c herein will be determined later, but it is always assumed that

$$0 \leqslant a < 1, \quad b < 1 \tag{1.3}$$

Inserting (1.2) into the dynamic equations of shell theory, after cancellations and manipulations based on using (1.1), we obtain:

the equilibrium equations

$$\frac{1}{A_i}\frac{\partial T_i^{\cdot}}{\partial \xi_i} + \frac{1}{A_j}\frac{\partial S^{\cdot}}{\partial \xi_j} + \eta^a \frac{p}{A_iA_j}\left[\frac{\partial A_j}{\partial \alpha_i}(T_i^{\cdot} - T_j^{\cdot}) + 2\frac{\partial A_i}{\partial \alpha_j}S^{\cdot}\right] - \eta^{2-a-c}p_7\left(\frac{N_i^{\cdot}}{R_{ii}} - \frac{N_j^{\cdot}}{R_{ij}}\right) - \frac{2Eh}{1-\sigma^2}p_1\eta^{2a-2b}d_\tau^2 u_i^{\cdot} = 0 \tag{1.4}$$

$$p_2\left(\frac{T_1^{\cdot}}{R_{11}} - \frac{2S^{\cdot}}{R_{12}} + \frac{T_2^{\cdot}}{R_{22}}\right) + \eta^{2-3a-c}p_3\left(\frac{1}{A_1}\frac{\partial N_1^{\cdot}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial N_2^{\cdot}}{\partial \xi_2}\right) + \eta^{2-2a-c}\frac{p_6}{A_1A_2}\left(\frac{\partial A_2}{\partial \alpha_1}N_1^{\cdot} + \frac{\partial A_1}{\partial \alpha_2}N_2^{\cdot}\right) - \frac{2Eh}{1-\sigma^2}\eta^{a-c-2b}p_5 d_\tau^2 w^{\cdot} = 0 \tag{1.5}$$

$$\frac{1}{A_i}\frac{\partial G_i^{\cdot}}{\partial \xi_i} - \frac{1}{A_j}\frac{\partial H^{\cdot}}{\partial \xi_j} + \eta^a \frac{p}{A_iA_j}\left[\frac{\partial A_j}{\partial \alpha_i}(G_i^{\cdot} - G_j^{\cdot}) - 2\frac{\partial A_i}{\partial \alpha_j}H^{\cdot}\right] - N_i^{\cdot} = 0 \tag{1.6}$$

the elasticity relationships

$$T_i^{\cdot} = \frac{2Eh}{1-\sigma^2}(\varepsilon_i^{\cdot} + \sigma\varepsilon_j^{\cdot}), \qquad S^{\cdot} = \frac{2Eh}{1+\sigma}\frac{\omega}{2} \tag{1.7}$$

$$G_i^{\cdot} = -\frac{2Ehr^2}{3(1-\sigma^2)}(\varkappa_i^{\cdot} + \sigma\varkappa_j^{\cdot}), \qquad H^{\cdot} = \frac{2Ehr^2}{3(1+\sigma)}\tau \tag{1.8}$$

the strain-displacement formulas

$$\varepsilon_i^{\cdot} = \frac{1}{A_i}\frac{\partial u_i^{\cdot}}{\partial \xi_i} + \eta^a \frac{p}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}u_j^{\cdot} - p_4\eta^{a-c}\frac{w^{\cdot}}{R_{ii}} \tag{1.9}$$

$$\omega^{\cdot} = \frac{1}{A_2}\frac{\partial u_1^{\cdot}}{\partial \xi_2} + \frac{1}{A_1}\frac{\partial u_2^{\cdot}}{\partial \xi_1} - \eta^a \frac{p}{A_1A_2}\left(\frac{\partial A_1}{\partial \alpha_2}u_1^{\cdot} + \frac{\partial A_2}{\partial \alpha_1}u_2^{\cdot}\right) + p_4\eta^{a-c}\frac{2w^{\cdot}}{R_{12}}$$

$$\varkappa_i^{\cdot} = -\frac{1}{A_i}\frac{\partial \gamma_i^{\cdot}}{\partial \xi_i} - \eta^a \frac{p'}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}\gamma_j^{\cdot} + \eta^{a+c}p'\frac{\delta^{\cdot}}{R_{12}} \tag{1.10}$$

$$\tau^{\cdot} = \frac{1}{A_1}\frac{\partial \gamma_2^{\cdot}}{\partial \xi_1} - \eta^a \frac{p'}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_1^{\cdot} + \eta^{a+c}p'\left(\frac{\omega_2^{\cdot}}{R_{11}} - \frac{\varepsilon_2^{\cdot}}{R_{12}}\right)$$

$$\gamma_i^{\cdot} = -\frac{1}{A_i}\frac{\partial w^{\cdot}}{\partial \xi_i} - \eta^{c+a}p'\left(\frac{u_i^{\cdot}}{R_{ii}} - \frac{u_j^{\cdot}}{R_{ij}}\right) \tag{1.11}$$

$$\omega_i^{\cdot} = \frac{1}{A_i}\frac{\partial u_j}{\partial \xi_i} - p\frac{\eta^a}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}u_i^{\cdot} + \eta^{a-c}p_4\frac{w^{\cdot}}{R_{ij}}$$

$$\delta^{\cdot} = \frac{1}{2}\left[\frac{1}{A_2}\frac{\partial u_1^{\cdot}}{\partial \xi_2} - \frac{1}{A_1}\frac{\partial u_2^{\cdot}}{\partial \xi_1} + p\frac{\eta^a}{A_1A_2}\left(\frac{\partial A_1}{\partial \alpha_2}u_1^{\cdot} - \frac{\partial A_2}{\partial \alpha_1}u_2^{\cdot}\right)\right]$$

It is assumed in the equations written above and everywhere henceforth, without any further reminder, that the subscripts i, j can take on the two pairs of values $i = 1, j = 2$

and $i = 2, j = 1$. Hence, each equation containing such subscripts must be considered as two; numerical subscripts are used in the single equations.

For convenience in the subsequent exposition, the provisional factors p, p_1', $p_1 - p_7$ are introduced into (1.4)–(1.11), and they must as yet be considered equal to unity. The symbol d_τ denotes differentiation with respect to the variable mentioned below.

Below we shall consider only such integrals of the dynamic shell theory equations for which the following assumption is satisfied: all the quantities marked by dots, together with their derivatives with respect to ξ_1, ξ_2, τ, have the same asymptotic behavior, i.e. are commensurate to the identical power of η. It is postulated that such integrals exist in definite domains of variation of α_1, α_2, t. The first two equalities in (1.2) show that the desired quantities acquire a factor η^{-a} for each differentiation (with respect to α_1, α_2) in the integrals possessing the formulated property, and the factor η^{-b} for each differentiation with respect to t (only the factors influencing the asymptotics are taken into account, i.e. which are powers of η). It hence follows that a is the index of variability of the desired state of stress-strain in the variables α_1, α_2, and b is the index of variability in t, which is later called the index of "dynamicity". The relative asymptotics of the desired quantities is given by the remaining equalities in (1.2). Thus, for example, the tenth and fourteenth equalities of (1.2) show that

$$T_i = BG_i, \qquad B = O\left(\eta^{2-3a+c}\right)$$

Within the scope of the assumptions made, the asymptotic order of each member in (1.4) – (1.11) is determined by the same powers of η which are written down explicitly for them. On this basis, let us hence proceed as follows.

Let the numbers a, b, c be fixed or subject to definite inequalities. Then the principal members (containing the least power of η in the coefficients) are easily found in each of the equations (1.4) – (1.11) taken separately. Discarding all the remaining terms in (1.4) – (1.11), we obtain an equation which can be called the principal equation. In general, the principal system will be contradictory (for instance, the number of unknowns in the system itself or in one of its subsystems will be greater than the number of equations). Hence, the values of a, b, c must be subjected to some constraints so that they would become admissible in this sense. It is shown below that a certain number of domains of admissible values of a, b, c can be constructed, to each of which corresponds a noncontradictory principal system distinct from the others.

The form of the principal system evidently determines very essential properties of the integral. Hence, it is natural to distinguish these latter according to its appropriate principal system. This is the first (static) criterion for the proposed classification of the integrals of the dynamic equations. The discussions in Sects. 2 – 5 are based thereon.

2. Membrane integrals. They are understood to be solutions corresponding to the case when a, b, c are subjected to the following constraints:

$$0 \leqslant a < {}^1/_2, \qquad c = a, \qquad b \leqslant 0 \tag{2.1}$$

Examining the coefficients of (1.4) – (1.11), we remark that the terms containing p_3, p_6, p_7 have factors of positive powers of η because of (2.1), and the powers of η in the remaining members are nonnegative. Hence, the principal system can be obtained in the case under consideration by choosing the provisional factors as follows:

$$p_3 = p_6 = p_7 = 0, \qquad p_1 = p_2 = p_4 = p_5 = p = p' = 1 \tag{2.2}$$

The same scheme is used to derive the principal equations below without detailed expositions. Inserting (2.2) into (1.4) – (1.7), we obtained the desired principal system. Equations (1.4), (1.5), (1.7) and (1.9) therein are

$$\frac{1}{A_i}\frac{\partial T_i^{\cdot}}{\partial \xi_i} + \frac{1}{A_j}\frac{\partial S^{\cdot}}{\partial \xi_j} + \frac{\eta^a}{A_iA_j}\left[\frac{\partial A_j}{\partial \alpha_i}(T_i^{\cdot} - T_j^{\cdot}) + 2\frac{\partial A_i}{\partial \alpha_j}S^{\cdot}\right] - \frac{2Eh}{1-\sigma^2}\eta^{2a-2b}d_\tau^{\,2}u_i^{\cdot} = 0$$

$$\frac{T_1^{\cdot}}{R_{11}} - \frac{2S^{\cdot}}{R_{12}} + \frac{T_2^{\cdot}}{R_{22}} - \frac{2Eh}{1-\sigma^2}\eta^{-2b}d_\tau^{\,2}w^{\cdot} = 0 \tag{2.3}$$

$$T_i^{\cdot} = \frac{2Eh}{1-\sigma^2}(\varepsilon_i^{\cdot} + \sigma\varepsilon_j^{\cdot}), \qquad S^{\cdot} = \frac{Eh}{1+\sigma}\omega^{\cdot}$$

$$\varepsilon_i^{\cdot} = \frac{1}{A_i}\frac{\partial u_i^{\cdot}}{\partial \xi_i} + \frac{\eta^a}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}u_j^{\cdot} - \frac{w^{\cdot}}{R_{ii}}$$

$$\omega^{\cdot} = \frac{1}{A_2}\frac{\partial u_1^{\cdot}}{\partial \xi_2} + \frac{1}{A_1}\frac{\partial u_2^{\cdot}}{\partial \xi_1} - \frac{\eta^a}{A_1A_2}\left(\frac{\partial A_1}{\partial \alpha_2}u_1^{\cdot} + \frac{\partial A_2}{\partial \alpha_1}u_2^{\cdot}\right) + \frac{2w^{\cdot}}{R_{12}}$$

These equalities yield nine equations (the equalities containing the subscripts i, j are double) with the nine unknowns

$$T_1^{\cdot},\ S^{\cdot},\ T_2^{\cdot},\ \varepsilon_1^{\cdot},\ \omega^{\cdot},\ \varepsilon_2^{\cdot},\ u_1^{\cdot},\ u_2^{\cdot},\ w^{\cdot} \tag{2.4}$$

Let us call them the principal subsystem, and the quantities (2.4) the principal unknowns. There remain the unused equations (1.11), (1.10), (1.8), (1.6). Introducing them into the analysis in the order in which they are listed, we can express the remaining unknowns of shell theory in terms of the quantities (2.4) by using direct operations. It is easy to see that here such solutions for which the displacements, the tangential stress resultants, and the tangential strain components are the basic unknowns, are called the membrane integrals, and the principal subsystem is the system of dynamical membrane equations.

Note. The terms containing η^a and η^{2a-2b} in (2.3) are conserved since a can be zero. If $a > 0$, then they can also be discarded. In particular, this means that the tangential inertial forces in the membrane integrals can play an essential part only for low variability.

3. Bending integrals. As yet, they are understood to be solutions obtained when

$$a > {}^1/_2, \qquad c = a, \qquad b \leqslant 2a - 1 \tag{3.1}$$

Multiplying (1.5) by η^{3a+c-2} in this case, and reasoning further exactly as in Sect. 2, we obtain that the following selection of the provisional factors

$$p = p' = p_1 = p_2 = p_6 = p_7 = 0, \quad p_3 = p_5 = p_4 = 1 \tag{3.2}$$

corresponds to a passage to the principal system.

Let us examine (1.5), (1.6), (1.8), (1.10), (1.11) and inserting (3.2) therein, let us write them as follows (certain terms with the factors p, p' and p_6 are still retained):

$$\frac{1}{A_1}\frac{\partial N_1^{\cdot}}{\partial \xi_1}+\frac{1}{A_2}\frac{\partial N_2^{\cdot}}{\partial \xi_2}+\eta^a\frac{p_6}{A_1A_2}\left(\frac{\partial A_2}{\partial \alpha_1}N_1^{\cdot}+\frac{\partial A_1}{\partial \alpha_2}N_2^{\cdot}\right)-$$
$$\frac{2Eh}{1-\sigma^2}\eta^{4a-2b-2}d_2^{\tau}w^{\cdot}=0$$
$$\frac{1}{A_i}\frac{\partial G_i^{\cdot}}{\partial \xi_i}-\frac{1}{A_j}\frac{\partial H^{\cdot}}{\partial \xi_j}+\eta^a\frac{p}{A_iA_j}\left[\frac{\partial A_j}{\partial \alpha_i}(G_i^{\cdot}-G_j^{\cdot})-\right.$$
$$\left.2\frac{\partial A_i}{\partial \alpha_j}H^{\cdot}\right]-N_i^{\cdot}=0 \tag{3.3}$$
$$G_i^{\cdot}=-\frac{2Ehr^2}{3(1-\sigma^2)}(\varkappa_i^{\cdot}+\sigma\varkappa_j^{\cdot}),\qquad H^{\cdot}=\frac{2Ehr^2}{3(1+\sigma)}\tau^{\cdot}$$
$$\varkappa_i^{\cdot}=-\frac{1}{A_i}\frac{\partial \gamma_i^{\cdot}}{\partial \xi_i}-\eta^a\frac{p'}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}\gamma_j^{\cdot},\quad \gamma_i^{\cdot}=-\frac{1}{A_i}\frac{\partial w^{\cdot}}{\partial \xi_i}$$
$$\tau^{\cdot}=\frac{1}{A_1}\frac{\partial \gamma_2^{\cdot}}{\partial \xi_1}-\eta^a\frac{p'}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_1^{\cdot}$$

The equalities (3.3) form the principal subsystem. They contain eleven equations with eleven principal unknowns

$$w^{\cdot},\ \gamma_1^{\cdot},\ \gamma_2^{\cdot},\ \varkappa_1^{\cdot},\ \varkappa_2^{\cdot},\ \tau^{\cdot},\ G_1^{\cdot},\ H^{\cdot},\ G_2^{\cdot},\ N_1^{\cdot},\ N_2^{\cdot} \tag{3.4}$$

Equations (1.4), (1.7), (1.9) and (1.11) remain to determine the unknowns not appearing in (3.4). They form a system of differential equations in these quantities about which more is said in Sect. 4.

It is henceforth considered that the bending integrals exist not only within the scope of the constraints (3.1) but also under less stringent conditions

$$a>{}^1/_2,\qquad c=a,\qquad (b \text{ is arbitrary}) \tag{3.5}$$

Then for sufficiently large b, the inertial terms in (1.4) and the first equation of (3.3) will contain η to negative powers, but it is assumed here and below that the appearance of a factor with negative power of η in the inertial (and only the inertial) terms is admissible (this will still be discussed in greater detail). If p, p', p_6 in (3.3) are taken equal to one (in place of zero), then these equations agree outwardly with the dynamic plate bending equations (in an arbitrary orthogonal curvilinear coordinate system). However, this agreement is not at all complete: the shell metric is generally different from the plate metric, and besides, p, p', p_6 must be considered zero in (3.3).

4. Planar integrals. These are understood to be solutions of the dynamic equations of shell theory obtained when a, b, c are subject to one of the following three groups of inequalities:

$$0\leqslant a<{}^1/_2,\qquad c<a,\qquad b>0 \tag{4.1^a}$$
$$a\geqslant{}^1/_2,\qquad c<a,\qquad b>2a-1 \tag{4.1^b}$$
$$a>{}^1/_2,\qquad c<a,\qquad b<2a-1. \tag{4.1^c}$$

Eliminating (1.5), (1.6), (1.8), (1.10) and (1.11) from consideration, we conclude that if any of the versions of condition (4.1) is satisfied, then it is necessary to set

$$p_4=p_7=0,\qquad p=p_1=1 \tag{4.2}$$

in order to go over to the principal system (the provisional factors p', p_2, p_3, p_5, p_6

enter only in the discarded equations and still remain undetermined). Equations (1.4), (1.7), (1.9) form the principal subsystem. By virtue of (4.2) it is written as follows:

$$\begin{gathered}\frac{1}{A_i}\frac{\partial T_i^{\bullet}}{\partial \xi_i}+\frac{1}{A_j}\frac{\partial S^{\bullet}}{\partial \xi_j}+\eta^a\frac{p}{A_iA_j}\left[\frac{\partial A_j}{\partial \alpha_i}(T_i^{\bullet}-T_j^{\bullet})+2\frac{\partial A_i}{\partial \alpha_j}S^{\bullet}\right]-\\ \frac{2Eh}{1-\sigma^2}\eta^{2a-2b}d_\tau^{\,2}u_i^{\bullet}=0\\ T_i^{\bullet}=\frac{2Eh}{1-\sigma^2}(\varepsilon_i^{\bullet}+\sigma\varepsilon_j^{\bullet}),\qquad S^{\bullet}=\frac{Eh}{1+\sigma}\omega^{\bullet}\\ \varepsilon_i^{\bullet}=\frac{1}{A_i}\frac{\partial u_i^{\bullet}}{\partial \xi_i}+\eta^a\frac{p}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}u_j^{\bullet}\\ \omega^{\bullet}=\frac{1}{A_2}\frac{\partial u_1^{\bullet}}{\partial \xi_2}+\frac{1}{A_1}\frac{\partial u_2^{\bullet}}{\partial \xi_1}-\eta^a\frac{p}{A_1A_2}\left(\frac{\partial A_1}{\partial \alpha_2}u_1^{\bullet}+\frac{\partial A_2}{\partial \alpha_1}u_1^{\bullet}\right)\end{gathered}\tag{4.3}$$

and consists of eight equations with eight unknowns

$$T_1^{\bullet},\ S^{\bullet},\ T_2^{\bullet},\ \varepsilon_1^{\bullet},\ \omega^{\bullet},\ \varepsilon_2^{\bullet},\ u_1^{\bullet},\ u_2^{\bullet}.\tag{4.4}$$

Returning to the equations discarded earlier, let us examine two cases:

Case 1. Let the conditions (4.1^a) or (4.1^b) be satisfied, from which there results that two inequalities are valid

$$b>2a-1,\qquad b>0\tag{4.5}$$

Then it is necessary to define c by the equality

$$c=a-2b\tag{4.6}$$

The terms with p_2 and p_5 in (1.5) hence contain η to zero power, and the exponent η is positive for p_3 and p_6. Therefore, it is necessary to take the additional equalities

$$p_3=p_6=0,\qquad p_2=p_5=1$$

which mean that in the principal system Eq. (1.5) becomes

$$\frac{T_1^{\bullet}}{R_{11}}-\frac{2S^{\bullet}}{R_{12}}+\frac{T_2^{\bullet}}{R_{22}}-\frac{2Eh}{1-\sigma^2}\eta^{a-c-2b}d_\tau^{\,2}w^{\bullet}=0\tag{4.7}$$

It permits determination of w since $T_1^{\bullet}$, $S^{\bullet}$, $T_2^{\bullet}$ enter into the group of principal unknowns (4.4) and can be considered as given. Equations (1.11), (1.10), (1.8), (1.6) remain unused. Introducing them into the analysis in the order in which they are listed, the remaining unknowns can be found by direct operations

$$\varkappa_1^{\bullet},\ \tau^{\bullet},\ \varkappa_2^{\bullet},\ G_1^{\bullet},\ H^{\bullet},\ G_2^{\bullet},\ N_1^{\bullet},\ N_2^{\bullet},\ \gamma_1^{\bullet},\ \gamma_2^{\bullet},\ \delta^{\bullet}\tag{4.8}$$

Case 2. Let there be compliance with the inequality

$$b<2a-1\tag{4.9}$$

corresponding to the requirements (4.1^c). Then in place of (4.5) it is necessary to take the following equality:

$$c=2-3a\tag{4.10}$$

(it is easy to see that (4.6) and (4.10) do not contradict the inequality $c<a$). An analysis of the coefficients in (1.5), (1.6), (1.8), (1.10), (1.11) hence shows that in addition

to (4.2) it is necessary to set

$$p_6 = p_5 = p' = 0, \; p_2 = p_3 = 1 \tag{4.11}$$

Consequently, Eq. (1.5) becomes in the principal system

$$\frac{1}{A_1}\frac{\partial N_1^{\bullet}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial N_2^{\bullet}}{\partial \xi_2} = -\frac{T_1^{\bullet}}{R_{11}^{\bullet}} + \frac{2S^{\bullet}}{R_{12}^{\bullet}} - \frac{T_2^{\bullet}}{R_{22}^{\bullet}} \tag{4.12}$$

Assumimg that the principal subsystem has already been solved, the right side of the equality therein must be considered known. The transverse stress resultants $N_1^{\bullet}$, $N_2^{\bullet}$ in the left side of (4.12) can be expressed in terms of w by using (1.6), (1.8), (1.10). Consequently, (4.12) goes over into an equation for the deflection $w^{\bullet}$. The remaining unknowns (4.8) are found in an obvious manner by using direct operations.

The system (4.3) agrees outwardly with the dynamic equations of the plane problem of elasticity theory, written in an arbitrary orthogonal curvilinear coordinate system. Below (3.3) and (4.3) are provisionally called, for simplicity, the bending problem and the plane problem equations, respectively. Later it will be necessary to distinguish the planar integrals depending upon as to which version of the requirements (4.1) they correspond. If it is requirements (4.1^a) or (4.1^b), then we shall speak of the planar integrals of the type $(c = a - 2b)$, otherwise, about the planar integrals of the type $(c = 2 - 3a)$. In solving the principal system corresponding to the planar integral of type $(c = a - 2b)$, the differential equations (with respect to α_1, α_2), namely, the equations of the plane problem (4.3), must be integrated only to construct the principal unknowns (4.4). The deflection w is afterwards found from (4.7) by integration with respect to τ, and the construction of the remaining unknowns is achieved by direct operations. It has been shown in Sect. 2 that the same situation holds also for the membrane integrals: only the membrane equations (2.3) need be integrated in solving the principal system. It may therefore be considered that the planar integrals of type $(c = a - 2b)$ are equivalent to the membrane integrals in the sense that they contain the identical number of arbitrarinesses of integration: these latter originate, respectively, during solution of (3.3) and (2.3) which are systems of identical order.

In solving the principal system corresponding to planar integrals of the type $(c = 2 - 3a)$, the system of differential equations must be integrated twice. The principal unknowns (4.4) are also determined from the equations of the plane problem (4.3) and w and the quantities (4.8) must be found from (4.12), (1.6), (1.8), (1.10), (1.11). They differ from (3.3) just in that the trinomial written on the right enters into (4.12) instead of the inertia term. Considering the equations of the plane problem solved, this expression must be considered known. Therefore, the inhomogeneous static plate bending equations comprise the second system mentioned above. In the sense of the number of arbitrary rules a planar integral of type $(c = 2 - 3a)$ is equivalent to the bending integral. For the latter (see Sect. 3) the principal unknowns (3.4) are determined from the bending equations (3.3), and the equalities (1.4), (1.7), (1.9) were indicated for the construction of the remaining unknowns in Sect. 3. Considering the quantities (3.4) therein as known, we see that these equalities are the inhomogeneous equations of the plane problem. Therefore, arbitrary rules originating both in the solution of equations of the plane problem and in the solution of the bending equations are contained in the planar integrals of type $(c = 2 - 3a)$ and in the bending integrals.

5. Bending-planar integrals. These are the solutions in which

$$a = {}^1/_2, \quad c = a, \quad b \leqslant 0 \tag{5.1}$$

For such values of a, b, c it is necessary to set

$$p = p' = p_1 = p_6 = p_7 = 0, \qquad p_2 = p_3 = p_4 = p_5 = 1 \tag{5.2}$$

in order to go over to the principal system. It is easy to see that the dissapearance of the provisional factors p, p', p_6, p_7 corresponds to the following simplifications: (1) discarding the antiderivatives of the desired functions (everywhere) as compared with their derivatives in comparison with α_1, α_2; (2) discarding of the transverse stress resultants in the equilibrium equations (1.4); (3) discarding members with u_1, u_2 in the formulas for $\varkappa_1$, τ, $\varkappa_2$. Therefore, (5.2) denotes the hypotheses taken for the theory of states of stress with high variability, and therefore, the principal system of the planar-bending integrals is the dynamic analog of the equations of this theory. They are known and are not presented here. Let us note that integrals corresponding to the simple edge effect are also contained, as particular cases, among the bending-planar integrals.

6. The classification of the integrals of dynamic equations according to the static criterion formulated at the end of Sect. 1 has been completed. The value of the number a, characterizing the variability of the desired quantities over the space coordinates, and the value of the number c characterizing the intensity of the normal deflections as compared with the intensity of the tangential displacements were decisive therein. The number b which governs the degree of dynamicity of the phenomenon studied hence remains almost arbitrary.

Having noted this, let us introduce the number ρ, equal to the least power of η, in the inertial terms of the principal subsystem, and let the value of ρ underlie the second (dynamic) criterion for classification of the integrals of the dynamic equations. Namely, let us call the integral quasi-static for $\rho > 0$, dynamic for $\rho = 0$, and strongly dynamic for $\rho < 0$. Each integral referred to a definite type according to the static criterion can, in turn, be classified according to the dynamic criterion also. The sense of ρ will hence be distinct for integrals of different type (according to the static criterion). Once again examining (2.3), (3.3) and (4.3), we see that for the membrane and bending-planar integrals

$$\rho = -2b \tag{6.1}$$

for the bending integrals

$$\rho = -2b + 4a - 2 \tag{6.2}$$

for the planar integrals

$$\rho = 2a - 2b \tag{6.3}$$

The double classification proposed is expedient. The static criterion directly evinces those approximate equations for an integral of given type, from which it can be determined (these are the principal equations). The mathematical specifics associated with the classification according to the dynamic criterion is discussed in Sect. 10.

7. Let us compare the proposed classification with the classification used in shell statics.

The passage from dynamics to statics can be accomplished by setting $b = -\infty$. In this case all those integrals for which no lower bound has been imposed on the value of b

remain valid. Among them are the membrane, bending, bending-planar integrals and planar integrals of the type ($c = 2 - 3a$). In combination they possess sufficient completeness so that the solution of any boundary value problem of shell statics could be complied therefrom if only the shell middle surface is not singular and all its edges are not asymptotic. This derives from the results elucidated in [1], say.

Indeed, if the variability of the desired stress-strain state is small ($a < {}^1/_2$), then the method of partition can be used for an approximate computation, i.e. the solution can be composed of membrane integrals and bending-planar integrals corresponding to simple edge effects. If $a = {}^1/_2$ the approximate computation can be performed by using the theory of states of stress with high variability. This means that the complete solution of the problem is determined by bending-planar integrals. For $a > {}^1/_2$ the partition of the state of stress into bending and tangential stress states occurs (see [1], Ch. 14, Sect. 15), i.e. the solution is a combination of bending and planar integrals of the type ($c = 2 - 3a$).

The case when the shell edges pass partially or entirely along the asymptotic lines of the middle surface is not included in the present analysis since the circumstances associated with the disappearance of normal curvature were not taken into account therein. Consequently, the generalized edge effects dropped out of the considerations. In order to include those cases, it is necessary to give a more flexible definition of the bending-planar integrals by replacing the equality $a = {}^1/_2$ therein by the inequality $a \leqslant {}^1/_2$. The length of the paper does not permit a more detailed examination.

Note. Only the planar integrals of type ($c = a - 2b$) become meaningless for $b = -\infty$. Such integrals are impossible in statics because the deflection w for them must be found from (4.7), and if w is independent of t, this operation is unrealizable.

8. Let us consider the free, steady vibrations of a shell. In such problems it may be considered that all the desired quantities depend only on α_1, α_2 and that the symbol of differentiation with respect to t is replaced by the factor $i\omega$ (ω is the frequency). By assumption (see Sect. 1), the asymptotics of the desired quantities should not change during differentiation with respect to the variable τ related to t by the second equality in (1.2). The $1 - \sigma^2$ there, differs slightly from unity, hence, the asymptotic relationship

$$\mu = \sqrt{\frac{m}{2Eh}}\,\omega = \eta^{-b} O(1) \tag{8.1}$$

must be satisfied. This equality introduces the frequency parameter μ and establishes its order relative to the quantity η^{-1} to be equal to the dynamicity index b. The asymptotic properties of the frequency parameter λ equal to μ^2 have been investigated in [2]. The deductions obtained are formulated in the terminology of the present paper as follows:

(1). There exist quasi-transverse vibrations. The shell deformation occurs therein mainly because of the displacements w with which the principal inertia terms are also associated.

(1^a). If $0 < a < {}^1/_2$, the vibrations are called quasi-transverse with low variability. They can be constructed by combining the membrane integrals with the bending planar integrals corresponding to the edge effects. The fundamental equations of the problem are hence the membrane equations (2.3). They govern the principal properties of the vibrations and, in particular, the asymptotics of the frequency parameter

$$\mu = \eta^{-b} \; O(1) = O(\eta^0) \tag{8.2}$$

(1^b). If $1/2 < a < 1$, the vibrations are called quasi-transverse with high variability. They are described by the bending integrals, (3.3) are the fundamental equations in this case, and the asymptotics μ is

$$\mu = \eta^{-b}\, O(1) = O(\eta^{1-2a}) \tag{8.3}$$

(2). There also exist vibrations which are called quasi-tangential. The shell strain occurs therein mainly because of the tangential displacements u_1, u_2. Also connected with these latter are the principal inertial forces.

(2^a). If $0 < a < 1/2$, the quasi-tangential vibrations can be constructed by combining planar integrals of the type $(c = a - 2b)$ with bending integrals.

Note. The integrals in [2] which must be adjoined to the membrane or planar integrals so that all the boundary conditions could be satisfied are called complementary. They can be obtained both as a particular case of bending-planar integrals and a particular case of bending integrals. The former of these possibilities has been used in case (1^a), and the latter in case (2^a).

(2^b). If $1/2 < a < 1$, the quasi-tangential vibrations are described by planar integrals of the type $(c = 2 - 3a)$.

The equations of the plane problem (4.3) are fundamental for the cases (2^a) and (2^b), and we have for the asymptotics of the frequency parameter

$$\mu = \eta^{b}\, O(1) = O(\eta^{a}) \tag{8.4}$$

In the limit case when $a = 0$, the difference between the quasi-tangential and quasi-transverse vibrations vanishes. Either can be constructed approximately by the scheme (1^a), and the asymptotics μ is determined by the relationships (8.2) or (8.4). In the other limit case, when $a = 1/2$, the quasi-transverse vibrations are described by bending-planar integrals, and the quasi-tangential vibrations by planar integrals of the type $(c = 2 - 3a)$. The asymptotics μ is hence determined by (8.3) and (8.4), respectively.

It has also been shown in [2] that vibrations with the ultralow frequencies $(\mu \ll 1)$ accompanied by the formation of a large number of nodal lines passing along the asymptotic lines of the middle surface are possible in shells of nonpositive curvature. It is impossible to detect them in the present investigation. This is also related to the fact that too rigorous requirement has been imposed on the number a in (5.1).

For given a formulas (8.2) – (8.4) permit determination of the index of dynamicity b for a given kind of vibration: for quasi-transverse vibrations $b = 0$ for $a \leqslant 1/2$ and $b = 1 - 2a$ for $a \geqslant 1/2$, while $b = a$ for quasi-tangential vibrations.

It also follows from the above that an integral of a specific type corresponds to each kind of vibration (if the frequencies are not ultralow) in the static classification, namely, the integral whose principal equations are fundamental for the vibrations under consideration. By means of (6.1) – (6.3) it is easy to verify that the values of b obtained by the method described, make ρ zero for the integrals corresponding to a given kind of vibrations. This permits giving the following physical interpretation to the second criterion for the classification proposed : membrane, bending, and planar integrals are quasi-static, dynamic, or strongly dynamic depending on whether the degree of their dynamicity is less than, equal to, or greater than the degree of dynamicity of the corresponding free vibrations.

9. If forced steady vibrations are studied, then the frequency ω introduced in Sect. 8 will be a given number. Therefore, the number b governing the index of dynamicity of the problem under consideration can be found by means of (8.1). The index of variability of the desired state of stress-strain a can also be considerd known (it can be assessed by means of the variability of the external effect [1, 3, 4]). This permits establishment in advance from what type of integrals the solution of the given problem should be composed, and therefore, pointing out the path to its approximate solution.

Example. Let a closed cantilever shell of revolution perform forced steady vibrations under the effect of tangential forces applied to the free edge, and varying according to the law $\sin n\varphi \sin \omega t$ (φ is the longitude). Then the index of variability of the external load θ and the degree of dynamicity b are determined from the equalities

$$n = \eta^{-\theta}, \qquad \sqrt{m/2Eh}\,\omega = \eta^{-b}$$

If $\theta < 1/2$, $b \leqslant 0$, then the solution of the problem can be composed from a membrane integral and a bending-planar integral corresponding to the simple edge effect (see Sect. 7). This means that in this case the partition method can be used as an approximate approach. There results from (6.1) that both the membrane and the bending-planar integrals are quasi-static for $b < 0$ and dynamic for $b = 0$.

If $\theta < 1/2$ and $b > 0$, then according to the last inequality in (3.1) and (5.1), the construction of the membrane and bending-planar integrals becomes impossible. The former must be replaced by the planar integral of the form $(c = a - 2b)$, and the latter by the bending integral. Therefore, for sufficiently high dynamicity $(b > 0)$ the method of partitions requires modification for investigation of steady forced vibrations: the fundamental state of stress must be determined from the Eqs. (4.3) of the plane problem (this denotes a diminution in the influence of the shell curvature), and the simple edge effect must be replaced by the state of stress corresponding to the bending integrals. It can be shown that the variability of the plane integral equals the variability of the external effect, i.e. $a = \theta$. Hence, it follows from (6.3) that the planar integral can be quasi-static (for $b < \theta$), dynamic (for $b = \theta$) and strongly dynamic (for $b > \theta$). A discussion of the bending integral from this viewpoint would require a great deal of space, so let us just note, without going into details, that the bending integral is always strongly dynamic in the problem under consideration.

10. The two-dimensional equations of thin shell theory contain a small factor in the coefficients of the highest derivatives (with respect to α_1, α_2). The general asymptotic theory of such equations has been developed in [5], where an important concept of regular degeneration is introduced. Such degeneration is characteristic for the boundary value problems of shell statics. All the most important approximate methods of static shell analysis can be interpreted as a consequence of the regularity of degeneration. In the dynamic problems, these properties are certainly conserved while only quasi-static integrals enter into the solution, since the static analysis can then be considered as the initial approximation of some iteration process.

In stationary problems (forced vibrations, say), nonregular degeneration is also possible. It will hold when the index of dynamicity is so great that dynamic and strongly dynamic integrals enter into the solution. Difficulties associated with resonance phenomena also occur at the same time. The case when a dynamic membrane integral enters into the solution is especially complex since transition lines γ generally appear in the domain under consideration, i.e. lines on which the type of dynamic membrane equations changes [6]. The assumption made in Sect. 1 about the asymptotics of solutions of the dynamic equations is incorrect in the neighborhood of γ and it is necessary to seek methods of merging the solutions on different sides of γ (this problem is solved in [7 – 9] for shells of revolution). The lines γ on which the type of equations changes are absent in the strongly dynamic case. Moreover, a large coefficient appears in the inertial terms. This opens the road to application of new asymptotic approaches and in the simplest cases (for example, in the analysis of shells of revolution) permits obtaining solutions in a perfectly elementary manner. However the attempts to generalize such approaches result in substantial difficulties connected with the nonregularity of degeneration.

Processes (elastic wave propagation, say) with inhomogeneous variability in time are studied in nonstationary dynamics problems. In this case the classification according to the dynamic criterion can be used to construct a (Lapalce) mapping. It can then be considered that $d_t^2 = p^2$ (p is the mapping parameter) and diverse approximate approaches can be applied in different ranges of variation in p. Difficulties associated with the need to construct the solution in a complex domain originate here. Meanwhile, new opportunities associated with the fact that degeneration is always regular in nonstationary problems, also occur.

In conclusion, let us note that N. A. Alumäa [10] first raised the questions discussed herein relative to nonstationary processes.

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DISTRIBUTION OF THE NATURAL FREQUENCIES OF A THIN ELASTIC SHELL OF ARBITRARY OUTLINE

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An asymptotic formula for the distribution function of the natural frequencies of a thin elastic shell is proved. The formula is used to determine the frequency density under different assumptions relative to the shell geometry. Density curves are presented.

1. Formulation of the problem. Fundamental results. The determination of the frequencies of a thin elastic shell clamped at the boundary results in the following eigenvalue problem (see [1], p. 97, [2], p. 297):

$$\sum_{j=1}^{3}\left(\frac{h^2}{12}n_{ij}+l_{ij}\right)u_j=\lambda u_i \qquad (i=1,\,2,\,3) \tag{1.1}$$

$$u_1\big|_\Gamma=u_2\big|_\Gamma=u_3\big|_\Gamma=\frac{\partial u_3}{\partial \nu}\bigg|_\Gamma=0 \tag{1.2}$$

Here u_i are components of the displacement vector of a point on the shell middle surface, l_{ij} and n_{ij} are the differential operators

$$\begin{aligned}
l_{11}u_1 &= -\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_1 - \\
&\quad \frac{1-\sigma}{2}\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_1-(1-\sigma)\frac{u_1}{R_1R_2}\\
l_{12}u_2 &= -\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_2+\frac{1-\sigma}{2}\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_2\\
l_{13}u_3 &= \frac{1}{A}\frac{\partial}{\partial\alpha}\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)u_3\right]-\frac{1-\sigma}{AR_2}\frac{\partial u_3}{\partial\alpha}\\
l_{21}u_1 &= -\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_1+\frac{1-\sigma}{2}\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_1\\
l_{22}u_2 &= -\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{AB}\frac{\partial}{\partial\beta}Au_2-\frac{1-\sigma}{2}\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{AB}\frac{\partial}{\partial\alpha}Bu_2-\frac{1-\sigma}{R_1R_2}u_2\\
l_{23}u_3 &= \frac{1}{B}\frac{\partial}{\partial\beta}\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)u_3\right]-\frac{1-\sigma}{BR_1}\frac{\partial u_3}{\partial\beta}\\
l_{31}u_1 &= -\frac{1}{AB}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\partial}{\partial\alpha}Bu_1+\frac{1-\sigma}{AB}\frac{\partial}{\partial\alpha}\left(\frac{B}{R_2}u_1\right)\\
l_{32}u_2 &= -\frac{1}{AB}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\partial}{\partial\beta}Au_2+\frac{1-\sigma}{AB}\frac{\partial}{\partial\beta}\left(\frac{A}{R_1}u_2\right)
\end{aligned} \tag{1.3}$$